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Two perspectives on a decohering spin*

Andreas Albrecht
NASA/Fermilab Astrophysics Center
P.O.B. 500
Batavia, IL 60510

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Abstract

I study the quantum mechanics of a spin interacting with an environment. Although the evolution of the whole system is unitary, the spin evolution is not. The system is chosen so that the spin exhibits loss of quantum coherence, or "wavefunction collapse", of the sort usually associated with a quantum measurement. The system is analyzed from the point of view of the spin density matrix (or "Schmidt paths"), and also using the consistent histories (or decoherence functional) approach.

1 Introduction

A cosmologist must face the issue of utilizing quantum mechanics without the benefit of an outside classical observer. By definition, there is nothing "outside" the universe! The traditional role of an outside classical observer is to cause "wavefunction collapse". This process causes a definite outcome of a quantum measurement to be realized, with the probability for a given outcome determined by the initial wavefunction of the system being measured. It is common to view this process as something that can not be

*Presented at the *Workshop on Time Asymmetry*, Sept 30 to October 4 1991, Masagón Spain. To appear in the proceedings published by Cambridge University Press, J.J. Halliwell, Editor described by a wavefunction evolving according to a Schrödinger equation, but which instead must be implemented "by hand".

There is a growing understanding that the essential features of wavefunction collapse *can* be present in systems whose evolution is entirely unitary. Pioneering work on this subject has been done by Zeh 1973, Zurek 1981, 1982, 1986, Joos and Zeh 1985, and Unruh and Zurek 1989, building on ideas of Everett 1957, and von Neumann 1955 The key is the inclusion of an "environment" or "apparatus" within the Hilbert space being studied. A subsystem can exhibit the non-unitary aspects of wavefunction collapse even though the system as a whole evolves unitarily. The wavefunction can then divide up into a number of different terms, each of which reflect a different "outcome". When there is negligible interference among the different terms during subsequent evolution, the "definiteness" of the outcome is realized in a restricted sense: Each term evolves as if the others were "not there", so a subsystem state within a given term evolves with "certainty" that its corresponding outcome is the only one. Nonetheless, the total wavefunction describes all possible outcomes, and one is never singled out.

In this work I study a two state "spin" system (subsystem 2) coupled to a 25-state "environment" or "apparatus" system (subsystem 1). The dynamics and the initial state are chosen to give the following behavior: The initial state is

$$|\psi_i\rangle = (a\,|\uparrow\rangle_2 + b\,|\downarrow\rangle_2) \otimes |X\rangle_1, \tag{1}$$

which evolves into the state

$$|\psi_f\rangle = a\,|\uparrow\rangle_2 \otimes |Y\rangle_1 + b\,|\downarrow\rangle_2 \otimes |Z\rangle_1 \tag{2}$$

Where $\langle Y|Z\rangle = 0$. This type of evolution is central to the standard way of describing a quantum measurement in the absence of an outside classical observer. Initially the spin subsystem is in a pure state, $(a\,|\uparrow\rangle_2 + b\,|\downarrow\rangle_2)$. For the state $|\psi_f\rangle$, the reduced density matrix of the spin $(\rho_2 \equiv \mathrm{tr}_1\,(|\psi_f\rangle\langle\psi_f|))$ has two non-zero eigenvalues, and the eigenstates are $|\uparrow\rangle$ and $|\downarrow\rangle$. The spin is no longer in a pure state, but may be said to be in $|\uparrow\rangle$ with probability a^*a and $|\downarrow\rangle$ with probability b^*b.

In $|\psi_f\rangle$ each of the spin states ($|\uparrow\rangle$ and $|\downarrow\rangle$) is uniquely correlated with its own state of the environment ($|Y\rangle_1$ and $|Z\rangle_1$ respectively) In this sense the environment has "measured" the spin. The two terms in Eq (2) represent the two possible "outcomes" of the measurement. The fact that initially the probability to find the spin in $|\uparrow\rangle$ or $|\downarrow\rangle$ is also a^*a or b^*b (respectively) illustrates that the measurement is "good": The probabilities of the different outcomes can be predicted from the initial wavefunction of the spin.

Another requirement of a good measurement is that $|\psi_f\rangle$ does not evolve back into the form of $|\psi_i\rangle$. This would amount to the environment "forgetting" the outcome of the measurement. This property is also well exhibited by the toy model studied here. Other features which are needed to match our intuitive notion of a good quantum measurement have to do with interactions of more than two subsystems. For example, one would want agreement among many observers that a particular outcome has been realized. Models can be constructed which exhibit this effect, but that is beyond the scope of this work.

The motivations for this work are twofold. The first goal is to develop some intuition as to what requirements one must place on the dynamics and initial state to produce the behavior just described. Secondly, I wish to explore the links between this approach and the "consistent histories" approach to the study of closed quantum systems (developed by Griffiths 1984, Omnes 1988, and Gell-Mann and Hartle 1990).

2 Results

The analysis presenteded here follows closely that of Albrecht 1992, and I refer you there for a description of the Hamiltonian and other details about the calculation. However, the results presented here are qualitatively different (see especially Section 6.3 of Albrecht 1992).

(For those interested in the technicalities, here are the differences from Albrecht 1992: The environment is larger, with $n_1 = 25$, and the couplings are $E_1 = .1, E_2 = .1$ and $E_I = 10$. Most importantly, the initial environment state is a coherent superposition of one eigenstate of $H^\uparrow$ and one eigenstate of $H^\downarrow$, in equal proportions. It is this difference which produces distinctively different behavior.)

Figure 1 shows information about the spin as the whole system evolves. Initially, the state is given by Eq (1), with $a = 0.7, b = 0.3$. In the lower plot, the solid curve gives p_1, the largest eigenvalue of ρ_2. It starts out at unity, as required by the "pure state" form of the initial conditions, and evolves to 0.7, where it holds steady. The dashed curve gives the entropy, S, of the spin ($S \equiv -\mathrm{tr}[\rho_2 log_2(\rho_2)]$), in units where the maximum possible entropy is unity. The entropy starts out zero and increases. This is always the case when a system evolves from a pure to a mixed state. (Note the the combined "spin $\otimes$ environment" system remains in a pure state, so *its* entropy is zero)

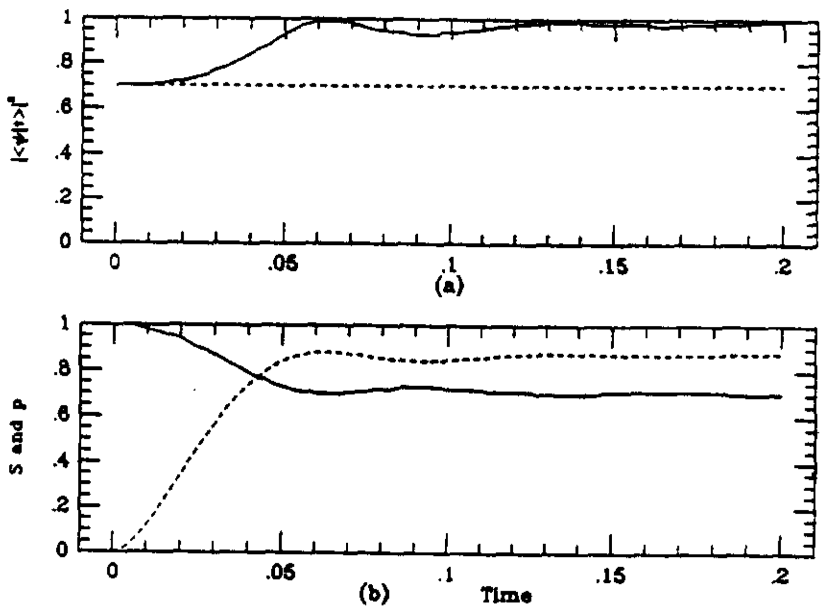

Figure 1: a: The solid curve is $|\langle\uparrow|1\rangle^S|^2$, and the dashed curve gives $\langle\uparrow|\rho_2|\uparrow\rangle$. b: The solid curve is the largest eigenvalue of ρ_2, the dashed curve is the entropy of the spin.

In the upper plot, the dashed curve gives the overall probability for the spin to be up, given by $\langle\uparrow|\rho_2|\uparrow\rangle$. This quantity is a "constant of the motion". The solid curve gives $|\langle\uparrow|1\rangle^S|^2$, where $|1\rangle^S$ is the eigenstate of ρ_2 (or "Schmidt state") corresponding to the largest eigenvalue. As discussed in Zeh 1973 and Albrecht 1992, the density matrix eigenstates correspond to a "Schmidt decomposition" of $|\psi\rangle$ (Schmidt 1907). When $|\psi\rangle$ is expanded in the eigenstates of ρ_2 and ρ_1, it always takes the form given by Eq (2), with each eigenstate of ρ_2 uniquely correlated with an eigenstate of ρ_1. This fact means the eigenstates of ρ_2 not only tell about ρ_2, but about the correlations with system 1 as well. The Schmidt states are said to trace out "Schmidt Paths" over time.

Since $|1\rangle^S$ belong to a two state Hilbert space, it is completely specified by $|\langle\uparrow|1\rangle^S|^2$, up to an overall phase. One can see that as the eigenvalue (p_1) approaches 0.7, the eigenvector becomes essentially $|\uparrow\rangle$. Thus the behavior promised in the previous section (Eqs (1) and (2)) is realized to a good accuracy.

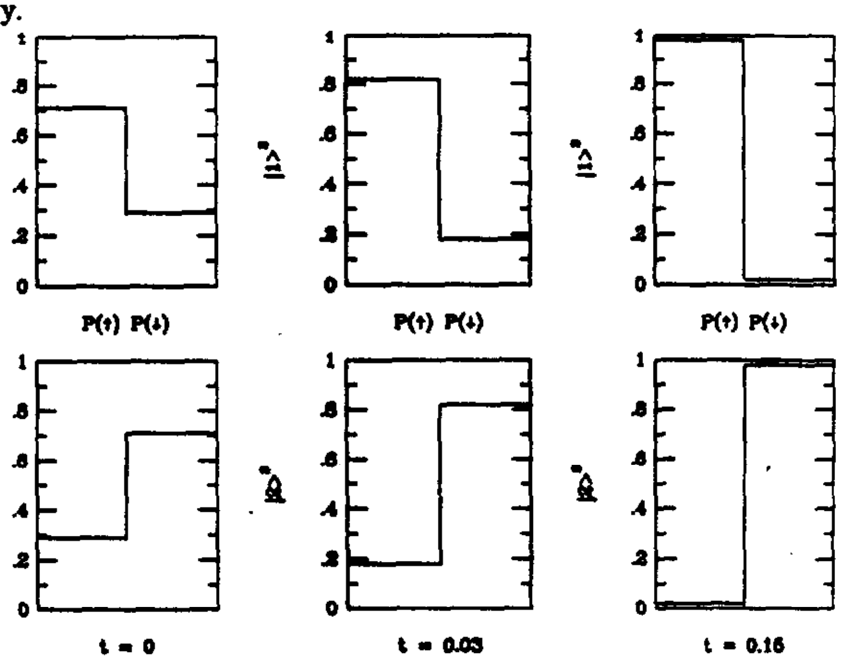

Figure 2: "A collapsing wavefunction." Each plot depicts an eigenstate of ρ_2 in terms of $p(\uparrow) \equiv |\langle\uparrow|i\rangle|^2$ and $p(\downarrow) \equiv |\langle\downarrow|i\rangle|^2$. The columns correspond to three different times. The two rows correspond to the two eigenstates.

Figure 2 is another representtation of the way the eigenstates of ρ_2 evolve. The first row represents $|1\rangle^S$, and the second row represents the other eigenvector. The three columns correspond to three times. The histogram in each plot provides two numbers, $p(\uparrow) \equiv |\langle\uparrow|1\rangle|^2$ and $p(\downarrow) \equiv |\langle\downarrow|1\rangle|^2$ for the first row, and similarly for the second eigenvector in the second row. In this way one can visualize a "collapsing wavefunction" by following the eigenstates of ρ_2.

3 Consistent Histories

I will now make contact with the "consistent histories" or "decoherence functional" approach to quantum mechanics of closed systems. Consider two projection operators:

$$\hat{P}_{\uparrow} \equiv |\uparrow\rangle\langle\uparrow| \otimes I_1; \quad \hat{P}_{\downarrow} \equiv |\downarrow\rangle\langle\downarrow| \otimes I_1 \tag{3}$$

where I_1 is the identity operator in the environment subspace, and $\{|\uparrow\rangle, |\downarrow\rangle\}$ form an orthonormal "projection basis" which spans the spin subspace. These projection operators sum to unity:

$$\hat{P}_{\uparrow} + \hat{P}_{\downarrow} = I. \tag{4}$$

One can take the formal expression for the time evolution:

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \tag{5}$$

and insert the unit operator $(\hat{P}_{\uparrow} + \hat{P}_{\downarrow})$ at will, resulting, for example, in the identity:

$$\begin{aligned}
|\psi(t)\rangle &= (\hat{P}_{\uparrow} + \hat{P}_{\downarrow})e^{-iH(t-t_1)}(\hat{P}_{\uparrow} + \hat{P}_{\downarrow})e^{-iHt_1}|\psi(0)\rangle & (6)\\
&= \hat{P}_{\uparrow}e^{-iH(t-t_1)}\hat{P}_{\uparrow}e^{-iHt_1}|\psi(0)\rangle + \hat{P}_{\uparrow}e^{-iH(t-t_1)}\hat{P}_{\downarrow}e^{-iHt_1}|\psi(0)\rangle & \\
&\quad + \hat{P}_{\downarrow}e^{-iH(t-t_1)}\hat{P}_{\uparrow}e^{-iHt_1}|\psi(0)\rangle + \hat{P}_{\downarrow}e^{-iH(t-t_1)}\hat{P}_{\downarrow}e^{-iHt_1}|\psi(0)\rangle & (7)\\
&\equiv |[\uparrow,\uparrow]\rangle + |[\uparrow,\downarrow]\rangle + |[\downarrow,\uparrow]\rangle + |[\downarrow,\downarrow]\rangle. & (8)
\end{aligned}$$

The last line just defines (term by term) a shorthand notation for the previous line. Each term represents a particular choice of projection at each time, and in that sense corresponds to a particular "path". In the path integral formulation of quantum mechanics the time between projections is taken arbitrarily small, and the time evolution is viewed as a sum over paths. For present purposes, the time intervals can remain finite, representing a "coarse graining" in time. Each term in the above expression is called a "path projected state", and the sum is a sum over coarse grained paths.

One attempts to assign the probability "$\langle [i,j] | [i,j] \rangle$" to the path $[i,j]$, but to make sense, the probabilities must obey certain sum rules. For example, one can define

$$|[\uparrow,\cdot]\rangle \equiv |[\uparrow,\uparrow]\rangle + |[\uparrow,\downarrow]\rangle, \tag{9}$$

where the "$\cdot$" signifies that *no* projection is made at t_1. One would want the probability for the path $[\uparrow,\cdot]$ to be the sum of the probabilities of the two paths of which it is composed:

$$\langle [\uparrow,\cdot] | [\uparrow,\cdot] \rangle = \langle [\uparrow,\uparrow] | [\uparrow,\uparrow] \rangle + \langle [\uparrow,\downarrow] | [\uparrow,\downarrow] \rangle \tag{10}$$

However, one can "square" Eq (9) to give the general result:

$$\langle [\uparrow,\cdot] | [\uparrow,\cdot] \rangle = \langle [\uparrow,\uparrow] | [\uparrow,\uparrow] \rangle + \langle [\uparrow,\downarrow] | [\uparrow,\downarrow] \rangle + \langle [\uparrow,\uparrow] | [\uparrow,\downarrow] \rangle + \langle [\uparrow,\downarrow] | [\uparrow,\uparrow] \rangle \tag{11}$$

Only if the last two terms in Eq (11) are small is the sum rule (Eq (10)) obeyed. When the relevant sum rules are obeyed the paths are said to give "consistent" or "decohering" histories. Advocates of this point of view argue that the only objects in quantum mechanics which make physical sense are sets of consistent histories. For a discussion of how this simple example links up with the (much more general) original work on this subject (Griffiths 1984, Omnes 1988, and Gell-Mann and Hartle 1990) , see Albrecht 1992.

4 Testing for consistent histories

Table 1a checks the probability sum rule (Eq (10)) for the toy model whose evolution is depicted in Fig 1. The projection times are $t_1 = .15, t = .2$, and the projection basis is $\{|\uparrow\rangle, |\downarrow\rangle\}$. The sum rule is obeyed to the accuracy shown. In fact, using the $\{|\uparrow\rangle, |\downarrow\rangle\}$ projection basis, the sum rule is obeyed no matter which projection times are chosen.

This result came as a surprise to me. After all the interesting behavior described in Figs 1 and 2, the consistent histories approach tells us that "$\uparrow$" and "$\downarrow$" paths are the right way to view the system, right through the period of "wavefunction collapse".

Consider for a moment a static (Hamiltonian = 0) spin, not coupled to any environment. It turns out that as long as the same projection basis

Table 1a path	value	Table 1b path	value
$\langle[\uparrow\uparrow]\|[\uparrow\uparrow]\rangle$	0.70	$\langle[\mathcal{I}\mathcal{I}]\|[\mathcal{I}\mathcal{I}]\rangle$	0.74
$\langle[\uparrow\downarrow]\|[\uparrow\downarrow]\rangle$	0.00	$\langle[\mathcal{I}\perp]\|[\mathcal{I}\perp]\rangle$	0.03
$\langle[\uparrow\cdot]\|[\uparrow\cdot]\rangle$	0.70	$\langle[\mathcal{I}\cdot]\|[\mathcal{I}\cdot]\rangle$	0.61
% violation	0%	% violation	25%

Table 1: Testing the probability sum rule (Eq (10)) for different paths. For 1a the sum rules are obeyed for any choice of t_1 and t. For 1b, $t_1 = .035$ and $t = 0.06$

is chosen at t and t_1, one always gets consistent histories. This is true for any projection basis. One could choose $\{|\uparrow\rangle, |\downarrow\rangle\}$ or one could choose the projection basis $\{|\mathcal{I}\rangle, |\perp\rangle\}$, where $|\mathcal{I}\rangle$ is the initial state of the spin $(a|\uparrow\rangle_2 + b|\downarrow\rangle_2)$, and $|\perp\rangle$ is the state orthogonal to it. A static spin would naturally result in unit probability for the $[\mathcal{I},\mathcal{I}]$ path, and zero probability for all other paths.

Table 1b shows the results for the fully interacting spin, using the $\{|\mathcal{I}\rangle, |\perp\rangle\}$ projection basis, but otherwise the same as Table 1a. Clearly the sum rules are not obeyed in this case. When I presented this talk, I felt that these results suggested the following link between the "setting up of correlations" described in Sections 1 and 2, and the consistent histories: The setting up of correlations tends to reduce number of different sets of consistent histories, and help one single out a preferred choice of projection basis.

Since then I have realized that things are not so simple. For one, the space of possible choices of projection basis is extremely large, even for the simple example discussed here. I have found the following new sets of consistent histories which are *not* consistent for the static spin. One chooses the projection basis at t_1 to be the eigenstates of ρ_2 at that time, and the projection basis at t to be $\{|\uparrow\rangle, |\downarrow\rangle\}$. These histories are consistent to the same accuracy as those shown in Table 1a, for any choice of t_1 and t.

For this article, I will not try to conclude anything about numbers of sets of consistent histories in the static versus interacting cases. I will simply remark that the special behavior of the correlations described by Eqs (1) and (2) does manifest itself in the consistent histories approach. It is interesting that this behavior does not show up in all consistent histories. After all, the $\{|\uparrow\rangle, |\downarrow\rangle\}$ projection basis generates consistent histories which are indistinguishable from those of an isolated static spin. With the inclusion of the interactions, however, there are new sets of consistent histories (such as those described in the previous paragraph) which do reflect the special evolution of ρ_2.

5 Consistent Histories vs Schmidt Paths

There is a view (well represented at this workshop) that says that consistent histories are the only physically correct objects to discuss in quantum mechanics. I am often asked if following the density matrix (or Schmidt paths) amounts to an alternative "interpretation". My present view is the following: The Schmidt analysis presented in the first part of this paper describes the evolution of correlations among subsystems. It is the point of view I am most familiar with. If I were to ask under what circumstances the evolution of correlations is physically interesting, there would be a number of requirements, such as stability or simplicity of evolution, which would come into play. Under these circumstances I suspect that the two approaches should be essentially equivalent.

For example, one could ask just what it means to follow the wavefunction through the collapse process, as described in Fig 1 and 2 of this paper. Surely it means nothing unless someone or something makes an observation and "catches it in the act". This would require interactions with a third system, presumably proceeding in a similar manner to the measurement depicted here. Such an interaction would have an impact on the consistent histories. I suspect, for example, that the $\{|\uparrow\rangle, |\downarrow\rangle\}$ would only remain a good projection basis during the collapse process if the new measurement was also made in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis.

In particular, I expect the equivalence of the two approaches to emerge from the role correlations among subsystems play in allowing the probability sum rules to be obeyed. (This has been elaborated to some degree in Albrecht 1992.) However, I still would like to better understand the relationship between the two approaches.

6 The arrow of time

As has been noted, for example by Zurek 1982 and Zeh 1971, 1990, there is an arrow of time built into the dynamics discussed here. This is dramatized in Fig 3, which is identical to Fig 1, but with the x-axis extended back to $t < 0$. One can seen that the pure "initial" ($t = 0$) state (which has zero entropy for the spin), is a very special state and the "collapse of the wavefunction" proceeds in the direction of increasing spin entropy. The $t < 0$ part of Fig 3 illustrates an "un-collapsing" wavefunction, where the correlations present between spin and environment at early times are lost, and the pure state emerges at $t = 0$. Then, for positive values of t correlations are established again. The stability of these correlations (and thus the goodness of the measurement) depend on another such "entropy dip", not occurring for $t > 0$. Even the simple system discussed here is complex enough for such entropy dips to occur very rarely.

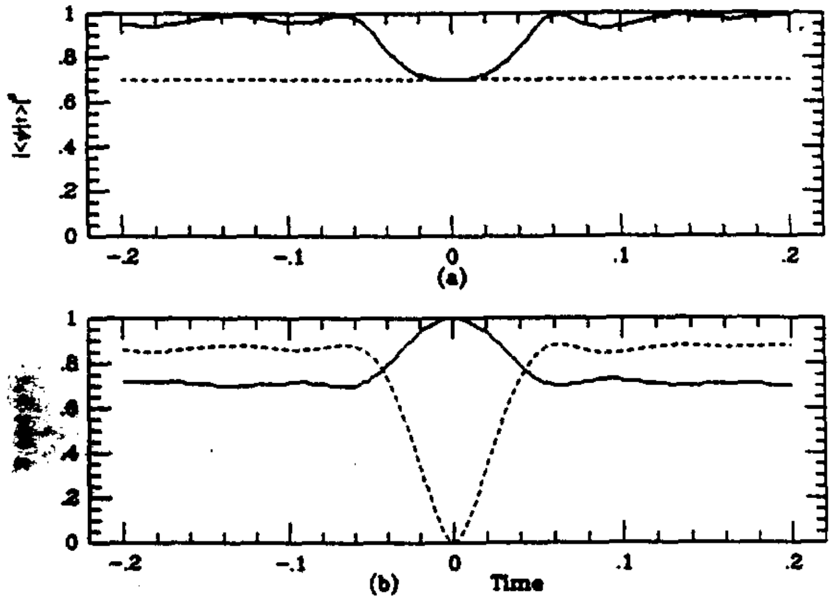

Figure 3: The same plots as Fig 1 extended back to $t = -2$.

Aside from questions of stability, how fundamentally is the arrow of time linked to quantum measurement? The initial state, $|\psi_i\rangle$ has zero entropy for the spin, so it is not surprising that just about anything will cause the entropy to increase. What about starting with a more general initial state? Schmidt tells us that (in a suitable basis) the most general state can be written

$$|\tilde{\psi}_i\rangle = \sqrt{p_1}|1\rangle_2 \otimes |1\rangle_1 + \sqrt{p_2}|2\rangle_2 \otimes |2\rangle_1. \quad (12)$$

It is simple to show that if one requires evolution which generalizes Eq (2) to give

$$\begin{aligned} |\tilde{\psi}_i\rangle &\rightarrow |\tilde{\psi}_f\rangle & (13)\\ &= \sqrt{p_1}\left(\langle\uparrow|1\rangle_2 |\uparrow\rangle \otimes |A\rangle_1 + \langle\downarrow|1\rangle_2 |\downarrow\rangle \otimes |B\rangle_1\right) & (14)\\ &\quad + \sqrt{p_2}\left(\langle\uparrow|2\rangle_2 |\uparrow\rangle \otimes |C\rangle_1 + \langle\downarrow|2\rangle_2 |\downarrow\rangle \otimes |D\rangle_1\right) & (15) \end{aligned}$$

then one must have increasing (or constant) entropy of the spin $(-\mathrm{tr}[\rho_2 \ln(\rho_2)])$ as $|\tilde{\psi}_i\rangle \rightarrow |\tilde{\psi}_f\rangle$. Thus "good measurement" appears to be closely linked with increasing entropy, even for high entropy initial states. (Note that I have chosen all four environment states, $|A\rangle_1$, $|B\rangle_1$, $|C\rangle_1$, and $|D\rangle_1$ to be mutually orthogonal. This means that in $|\tilde{\psi}_f\rangle$ the environment has a record of whether the spin is up or down, *and* which term of Eq (12) has been "measured".)

7 Conclusions

The ideas put forward by Zeh 1973, Zurek 1982, Joos and Zeh 1985, and Unruh and Zurek 1989, have sufficiently de-mystified the notion of wavefunction collapse that one can actually unitarily follow the evolution of a system right through the collapse process. I have investigated a simple system which exhibits "wavefunction collapse". I find Zeh's idea of watching the evolution of the eigenstates of the reduced density matrix particularly appealing. This approach allows one to follow exactly the evolution of the correlations among subsystems. It also allows one to visualize the collapse process quite explicitly, as illustrated in Fig 2.

I also applied the "consistent histories" analysis (of Griffiths 1984, Omnes 1988, and Gell-Mann and Hartle 1990) to the same system. This approach allows one to consider may different sets of histories for the system. In the example studied here, many different sets passed the consistency test. It is intriguing that one set of consistent histories for the spin did not reflect the interesting evolution of the correlations between the spin and the environment. That set of histories would look the same for a static spin, decoupled from the environment. Other consistent histories exhibited quite direct links to the "quantum measurement" process underway.

I have described a system which exhibits interesting behavior, both in terms of evolving correlations, and in terms of consistent histories. All the behavior discussed here lends itself to simple explanation in terms of the nature of the Hamiltonian and the initial state. Understanding this relationship is one of the main goals of this work, and it will be spelled out in a future publication.

8 Acknowledgements

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